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Some Fixed Point Theorems for Nonexpansive Mapping in Hilbert Space

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Abstract

In this paper we prove a fixed point theorem for nonexpansive mapping using a well known result of Ky Fan's best approximation theorem in Hilbert space setting.

AMS Subject Classification: 47H10, 54H25.

Keywords: Fixed point theorem, Ky Fan's best approximation theorem, Nonexpansive mapping.

Introduction

In 1965, Browder, Godhe and Kirk proved that each nonexpansive map $T: S \to S$, where S is a particular set in a Hilbert space having at least one fixed point. A mapping $T: X \to X$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in X$ .The fundamental properties of contraction mapping do not extend to nonexpansive mapping. It is of great importance in applications to find out if nonexpansive mappings have fixed points. Several interesting fixed point theorems have been proved by using Ky Fan's best approximation theorem. This approach helps to find fixed point theorems under different boundary conditions. Most of the fixed point theorems are given for self maps that are for a function with domain and range the same. In case a function does not have the same domain and range then we need a boundary condition to guarantee the existence of fixed point. In 1969, Ky Fan's [3] establish theorem known as Ky Fan's best approximation theorem which has been of great importance in approximation theory, nonlinear analysis and fixed point theory. In general, fixed point theorems and related technique have been used to prove the result about best approximation. We refer to A. Carbone [2], J. Li. and S.P. Singh [4], T.C. Lin [5], S. Park [8], R. Schoneberg [9], G. Marino et al. [6]. In this paper the concept of nonexpansive mapping has been used to establish a Ky Fan's best approximation theorem which generalized the results of some standard results on Hilbert space.

Preliminaries

Definition 2.1[7] Suppose that $T: H \to H$ be a map where H is a Hilbert space. Then T is said to be nonexpansive if

$$\|T(x) - T(y)\| \le \|x - y\| \ \ \forall \ x, y \in H .$$

Theorem2.2 [11] If S is closed bounded convex subset of R^n and $T: S \to R^n$ is a continuous function then there is a $z \in S$ such that $\langle Tz, x - z \rangle \ge 0, \ \forall \ x \in S$.

Theorem2.3 [11] Suppose that S be a closed convex subset of a Hilbert space H and $T: S \to H$ be a nonexpansive map with $T(S)$ bounded. Then there is a $z \in S$ such that $\|z - Tz\| = d(Tz, S)$.

Theorem2.4 [11] Suppose that S be a closed bounded convex subset of Hilbert space H and $T: S \to H$ be a monotone continuous map. Then there is a $z \in S$ such that $\langle Tz, x - z \rangle \ge 0, \forall x \in S.$

Definition2.5 [10] Suppose that X be a normed linear space and S be a nonempty subset of X. Suppose $x \in X$. An element $y \in S$ is called an element of best approximationto x , if $\|x - y\| = d(x, S) = \inf\{\|x - z\| : z \in S\}$.

Let $P(x)$ denote the set of all points in S closed to x that is

$$P(x) = \{y \in S : \|x - y\| = d(x, S)\}.$$

The set S is said to be proximinal if P(x) is nonempty for all $x \in X$.

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If P(x) is a singleton for each $x \in X$ then S is said to be Chebyshev set. If S is a Chebyshev set then P is a single valued map. In case $P: X \to 2^S$ is a multivalued map it is called the metric projection or best approximation operator.

Definition2.6 [1] Let S be a closed convex subset of a real Hilbert space H. Then for each $x \in H, x \notin S$ there is a unique $y \in S$ nearest to x that is,

$\|x-y\| = d(x,S) = \inf\{\|x-z\|, z \in S\}$, If S is a chebyshev set, and

$$P(x) = \{y \in S : \|x-y\| = d(x,S)\},$$

is the proximity map or best approximation operator on H.

Main Results

Theorem 3.1 Suppose S be a closed convex subset of a real Hilbert space H. Then for each $x \notin S$ there exists a unique $y \in S$ nearest to x, that is $\|x-y\| = d(x,S) = \inf\{\|x-u\| : u \in S\}$ In this case S If S is a chebyshev set, and

$$P_S(u) = \{v \in S : \|u-v\| = d(u,S)\}$$

is the best approximation operator or the proximity map on H.

Theorem3.2 The proximity map P satisfies the following property.

[1] $\left(u-Pv, Pu-Pv\right) \geq 0$, $\forall u,v \in H$,

[2] $\|Pu-Pv\| \leq \|u-v\|, \forall\ u,v \in H$,

[3] $\|u-Pu\|^2 + \|Pu-v\|^2 \leq \|u-v\|^2$, $\mathrm{v} \in \mathrm{S}, \mathrm{u} \notin \mathrm{S}$.

Proof:

[1] $(\mathrm{u-Pv, Pu-Pv}) \geq 0, \forall\, \mathrm{u, v} \in \mathrm{H},$ implies that

$\left(u, Pu-Pv\right) \geq (Pu, Pu-Pv),$

Similarly $\left(v-Pv, Pv-Pu\right) \geq 0$

Implies that $\left(v, Pv-Pu\right) \geq \left(Pv, Pv-Pu\right),$

That is $\left(-v, Pu-Pv\right) \geq \left(-Pv, Pu-Pv\right)$

Adding we get

$\left(Pu-Pv, u-v\right) \geq \left(Pu-Pv, Pu-Pv\right)$

This implies that $\|Pu-Pv\| \leq \|u-v\|$

Equality holds if $\|u-Pu\| = \|v-Pv\|$

[2] For $0 \leq \beta \leq 1$,

$$\|u-Pu\|^2 \leq \|u-\beta Pv-(1-\beta)Pu\|^2$$
$$= \|u-Pu+\beta(Pu-Pv)\|^2$$
$$= \|u-Pu\|^2 + \beta^2\|Pu-Pv\|^2 + 2\beta(u-Pu, Pu-Pv)$$

.

That is $\beta^2\|Pu-Pv\|^2 + 2\beta(u-Pu, Pu-Pv) \geq 0$.

This will be overruled for small β unless $\left(u-Pu, Pu-Pv\right) \geq 0$.

[3] For $u \in H$ and $v \in S$

$$\|u-Pu\|^2 + \|Pu-v\|^2 \leq \|u-Pu\|^2 + \|Pu-v\|^2 + 2(u-Pu, Pu-v)$$
$$= \|u-Pu+Pu-v\|^2$$
$$= \|u-v\|^2.$$

Hence proved.

Theorem3.3 Suppose that H be a Hilbert space $T: H \to H$ is a non expansive mapping and I is the identity map then $I-T$ is monotone, that is $\left\langle (I-T)u-(I-T)v, u-v\right\rangle \geq 0.$

We take $I-T=F.$ Then $T=I-F.$

Proof: Since T is nonexpansive,

$$\|Tu-Tv\|^2 \leq \|u-v\|^2$$
$$\leq \|u-v\|^2 + \|Fu-Fv\|^2$$

Now $\|(I-F)u-(I-F)v\|^2$

$$= \|u-v-(Fu-Fv)\|^2$$
$$= \|u-v\|^2 + \|Fu-Fv\|^2 - 2(Fu-Fv, u-v)$$
$$\leq \|u-v\|^2 + \|Fu-Fv\|^2,$$

only if $\left(Fu-Fv, u-v\right) \geq 0$

Hence F is monotone.

Theorem3.4 Suppose that H be a Hilbert space and S be a closed convex subset of H. Suppose $T: S \to H$ be a non expansive mapping with T(S) bounded. Then there exists a $v \in S$ such that $\|v-Tv\| = d(Tv, S)$.

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Proof: Suppose $P:H \to S$ be the proximity map .Then $PT:S \to S$ is nonexpansive . Suppose that $B=\overline{co(PT(S))}$ $(PT = P \circ T)$ .Then B is closed bounded and convex and $F = PT: B \to B.$ By Browder's fixed point theorem F has a fixed point. That is $PTu_0 = u_0,$ for some $u_0 \in B.$
Therefore $\|u_0 - Tu_0\| = d(Tu_0, S)$.

Corollary3.5 Let S be a closed bounded convex subset of H and $T:S \to H$ is a nonexpansive map then there is $v_0 \in S$ such that $\|v_0 - Tv_0\| = d(Tv_0, S)$.

Following are the fixed point theorems which are derived as corollaries:

Corollary3.5.1 Suppose B_r be a closed ball of radius r and centre 0 in a Hilbert space H. Let $T:B_r \to H$ be a nonexpansive map with the property that if $Tu = \beta u$ for some $u \in \partial B_r$, then $\beta \le 1.$ Then T has a fixed point.

Proof: $\exists\ v \in \partial B,$ such that $\|v - Tv\| = d(Tv, B_r)$. If $Tv \notin B_r$, then $v = PTv \in \partial B_r$, that is $\|v\| = r$, so

$$\beta = \frac{\|Tv\|}{\|v\|} = \frac{\|Tv\|}{r} > \frac{r}{r} = 1,$$ **a** contradiction.

Therefore $Tv \in B_r$ and T has a fixed point.

Corollary3.5.2 Suppose H be a Hilbert space and S be a closed bounded convex subset of H. Let $T:S \to H$ be nonexpansive. Assume forany $u \in \partial S$ with $u = PTu$ such that u is a fixed point of T. Then T has a fixed point.

Proof: Using corollary 3.5 $\exists\ v \in S$ such that $\|v - Tv\| = d(Tv, S).$ In case $Tv \in S$ then there is nothing to prove .If $Tv \notin S$ then $v = PTv \in \partial S$.
So we get hypothetically $v = Tv$.

Corollary3.5.3 Suppose that S be a nonempty closed convex subset of Hilbert space H and suppose $T:S \to H$ be a nonexpansive mapping such that there exists a bounded subset M of S satisfying the condition, for all $u \in S$ there exists a $v \in M$ such that $\|Tu - v\| \le \|u - v\|.$ Then T has a fixed point.

Proof: Suppose $B = \overline{co(M)}$ the convex closure of M. By corollary 3.5 we get a $v \in B$ such that $\|v - Tv\| = d(Tv, B)$.
If $T(v) \notin B$ then hypothetically
$\exists\ v_0 \in M$ such that $\|Tv - v_0\| \le \|v - v_0\|$.
But

$$\|PTv - Tv\|^2 + \|PTv - v_0\|^2 \le \|Tv - v_0\|^2 \le \|v - v_0\|^2$$

Implies $\|v - T(v)\|^2 + \|v - v_0\|^2 \le \|v - v_0\|^2$

Implies $\|v - T(v)\|^2 \le 0$ which gives $Tv = v$.

Corollary3.5.4 Suppose S be a closed convex subset of a Hilbert space H and $T:S \to H$ be a nonexpansive mapping .Let T(S) be bounded and $T(\partial S) \subset S.$ Then T has a fixed point.

Proof: Using theorem 3.4 there is $v_0 \in S,$
Such that $\|v_0 - Tv_0\| = d(\mathrm{Tv_0, S})$.
If $Tv_0 \in S$, then $v_0 = Tv_0$.
Otherwise $v_0 \in \partial S$ and since $T(\partial S) \subset \mathrm{S}, \mathrm{Tv_0} \in \mathrm{S}$
and hence $Tv_0 = v_0$.

Corollary3.5.5 Suppose S be a nonempty closed bounded convex subset of Hilbert space H and let $T:S \to H$ be a nonexpansive mapping such that for each $u \in \partial S$ $\|Tu - v\| \le \|u - v\|$ for some $v \in S.$ Then T has a fixed point.

Proof: There is $v \in S$ such that
$\|v - Tv\| = d(\mathrm{Tv, S}).$
Let $Tv \notin S.$ Then $v = PTv \in \partial S$.
Hypothetically $\|Tu - v\| \le \|v - u\|$
For some $u \in S$.
Also

$$\|Tv - PTv\|^2 + \|PTv - u\|^2 \le \|Tv - u\|^2 \le \|v - u\|^2$$

This satisfy only if $\|Tv - PTv\| = 0,$

Since $\|PTv - u\|^2 = \|v - u\|^2$

Hence $Tv = v$

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